Flow of a Liquid Film down a Slightly Inclined Tube

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The flow of thin liquid films down tubes is of interest in many chemical engineering applications and has been the subject of a considerable number of theoretical and experimental investigations. The object of this work is to examine an observation reported by Ponter and Davies (1966). Experiments were performed on the breakdown of films flowing under gravity on vertical tubes, and great difficulty was met in obtaining consistent results unless elaborate precautions were taken to ensure that the tube was accurately vertical. (A traveling microscope was used.) Qualitatively this is not surprising for obviously if the tube is sloping, the fluid will run round to the bottom, causing thinning of the upper portions of the film—a condition which is known to favor rupture of the film. The extremely small angles involved are difficult to explain. In the experiments reported, deviations of as little as 8′ from the vertical were sufficient to cause major changes in the flow. Elementary considerations suggest that an inclination to the vertical of α will have an effect in a downstream distance of $R \cot \alpha$, which is about 600 cm in the present case. However, the effects seem to be noticeable at one-tenth of this distance, raising doubts as to whether the numerical factors involved are sufficient to explain this away.

The importance of this phenomenon is not only in engineering practice where the tubes cannot be so accurately aligned with the vertical, but it may vitiate experimental investigations into other properties of the film.

AVERAGED EQUATIONS OF MOTION

Cylindrical polar coordinates (r, θ, z) are chosen as shown in Figure 1. The tube radius is R, the film thickness is $H(\theta, z)$, and the velocity components are (w, v, u).

The inertia terms in the equations of motion are neglected which is reasonable in most experimental situations. The viscous terms are approximated and averaged in the same manner as proposed by Lamb (1932). $\partial/\partial z$ and $1/r\ \partial/\partial\theta$ are assumed negligible compared with $\partial/\partial r$, so that the equations for u and v become

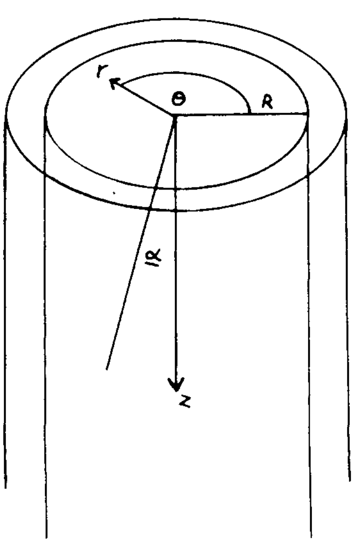

Fig. 1. Sketch of coordinate system.

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g \cos\alpha + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \quad (1)$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g \sin\alpha \sin\theta + \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \quad (2)$$

We put $u = C_1 f(r)\,\overline{u}(\theta, z)$ and $v = C_2\, g(r)\,\overline{v}(\theta, z)$, where f and g satisfy

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 1$$

$$\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{g}{r^2} = \frac{1}{r}$$

and $f(R) = g(R) = 0$, $f'(R+h) = g'(R+h) = 0$. This represents the no-slip condition on the tube and the zero shear-stress condition at the free surface. The "constants" C_1 and C_2 are chosen to make $\overline{u}$ and $\overline{v}$ the average velocities, that is,

$$\int_R^{R+h} r u\, dr = \int_R^{R+h} r\overline{u}\, dr$$

and

$$\int_R^{R+h} v\, dr = \int_R^{R+h} \overline{v}\, dr$$

This gives $C_1 = -3/h^2$, $C_2 = -3R/h^2$ with error of order h/R.

The condition of continuous normal stress at the free surface may be taken as $p = 0$, from which it follows that $p = 0$ everywhere. (There is a hydrostatic pressure gradient because the tube is inclined, but this has negligible effect). The neglect of surface tension at the free surface requires an explanation. The curvature of the free surface is about $(R + h)^{-1}$ and this produces a normal stress $T(R + h)^{-1} = T(1/R - h/R^2 \ldots)$ approximately. The first term is a constant and produces no motion since only pressure gradients enter the equations of motion. On the assumption that $\partial/\partial z$ and $1/r\ \partial/\partial\theta$ are about R^{-1} it is seen that surface tension produces pressure gradients of order Th/R^3 and the ratio of this to gravitational effects is $Th/\rho g R^3$. This number is typically about 10^{-3}.

The equations of motion now reduce to

$$0 = g \cos\alpha - \frac{3\nu\overline{u}}{h^2} \quad (3)$$

$$0 = g \sin\alpha \sin\theta - \frac{3\nu\overline{v}}{h^2} \quad (4)$$

The continuity equation is

$$\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial r}(rw) = 0 \quad (5)$$

and this may be combined with the kinematic free-surface condition

$$w = u_s \frac{\partial h}{\partial z} + \frac{v_s}{r} \frac{\partial h}{\partial \theta} \quad (6)$$

to give an average continuity equation. (The subscript s indicates evaluation at the free surface $r = R + h$.) Equation (5) when multiplied by r and integrated across the film, with u and v replaced by $\overline{u}$ and $\overline{v}$, together with (6) gives

$$h \frac{\partial \overline{u}}{\partial z} + \frac{3}{2} \overline{u} \frac{\partial h}{\partial z} + \frac{h}{R} \frac{\partial \overline{v}}{\partial \theta} + \frac{3}{2} \frac{\overline{v}}{R} \frac{\partial h}{\partial} = 0 \quad (7)$$

The factors of 3/2 enter since $u_s = 3/2\ \overline{u}$ etc., which follows from (1) and (2).

SOLUTION OF EQUATIONS OF MOTION

Equations (3) and (4) may be used to eliminate $\overline{u}$ and $\overline{v}$ from (6), leading to an equation for h:

$$R \cot \alpha \frac{\partial h}{\partial z} + \sin \theta \frac{\partial h}{\partial \theta} + \frac{2}{7} (\cos \theta) h = 0 \quad (8)$$

This is a linear first-order equation and may be solved by elementary methods to give

$$h(\sin \theta)^{2/7} = F(\eta) \quad (0 \leqq \theta \leqq \pi) \quad (9)$$

where F is an arbitrary function and

$$\eta = \cot \frac{\theta}{2} \exp \left(\frac{z \tan \alpha}{R} \right) \quad (10)$$

The arbitrary function F is determined by initial conditions and we may suppose that h is a given function of θ at $z = 0$. For the simple choice $h = h_0 =$ constant

$$F(\eta) = \left(\frac{2\eta}{1 + \eta^2} \right)^{2/7} h_0$$

and finally

$$\left(\frac{h}{h_0} \right)^{7/2} \sin \theta = \frac{2e^{\frac{z \tan \alpha}{R}} \cot \frac{\theta}{2}}{e^{\frac{2z \tan}{\alpha/R}} \cot^2 \frac{\theta}{2} + 1} \quad (11)$$

This solution shows some interesting properties. For each fixed θ, $h \rightarrow 0$ exponentially, in fact

$$\left(\frac{h}{h} \right)^{7/2} \sim \sec^2 \frac{\theta}{2} \exp \left(- \frac{z \tan \alpha}{R} \right) \quad (12)$$

This of course is not uniform in θ. The approximation fails near $\theta = \pi$ where

$$\left(\frac{h}{h_0} \right)^{7/2} \sim \operatorname{cosec}^2 \frac{\theta}{2} \exp \left(\frac{z \tan \alpha}{R} \right) \quad (13)$$

Thus the film thins exponentially over most of the tube, but thickens exponentially near the lowermost part (in fact, in a zone $(\pi - \phi, \pi + \phi)$ where ϕ is of order $\exp(-z \tan \alpha / R)$).

DISCUSSION

Ponter et al. (1967) have reported measurements of the minimum wetting rate on various types of tube. This is the minimum flow rate at which dry patches do not appear. Results are given for tubes of length 30 cm and 120 cm, other factors being kept constant, and these show almost without exception that the minimum wetting rate is larger for the longer tube by about 10 or 20%. This suggests strongly that an instability is present which has more time to grow on the longer tube, and one possibility is the local thinning described here.

To examine this possibility in more detail, since the wetting rate is proportional to h^3, then a 15% change in wetting rate is equivalent to a 5% change in h. The above observations can be explained if the film becomes 5% thinner on the upper portions in a distance 120 cm − 30 cm = 90 cm. Putting $\theta = 0$, $\alpha = 8'$, $R = 1.27$ cm, $z = 90$ cm into (11)

$$\frac{h}{h_0} = \exp \left(- \frac{2}{7} \frac{z \tan \alpha}{R} \right) = 0.96 \text{ approx.}$$

which is a good agreement.

Earlier experiments by Friedman and Miller (1941) are mentioned by Ponter et al. (1967). Here it was noticed that variations in $\overline{u}$ of 100 to 200% occurred as θ varied from 0 to 2π. This corresponds to variations in h of 50 to 100%. This cannot be explained with the small values of α reported, but it is possible that these large effects were found at somewhat larger angles. Friedman and Miller gave little detail about this aspect of their work, and presumably they were more concerned to eliminate this effect than to investigate it.

However, a further effect, a consequence of their method of putting the film on the tube, must be allowed for. The film flows down the inside of the tube and is produced by surrounding the upper portion of the tube by a reservoir and allowing water to flow over the rim. When the tube is inclined, not all portions of the rim will be the same distance below the level in the reservoir, and in fact the maximum difference will be $2R \sin \alpha$.

This situation may be modeled by supposing that at $z = 0$, $h_0(\theta) = H - R \sin \alpha \cos \theta$ where H is the average thickness. The corresponding solution of (9) is easily obtained, that is,

$$\frac{h_{\max}}{h_{\min}} = \frac{H + R \sin \alpha}{H - R \sin \alpha} \exp \left(\frac{4}{7} \frac{z \tan \alpha}{R} \right)$$

where $h_{\max}$ and $h_{\min}$ are the values of h at $\theta = \pi$ and $\theta = 0$, respectively. Putting

$$\alpha = \frac{1}{300}, \quad H = 4.2 \times 10^{-2} \text{ cm}, \quad R = 1.25 \text{ cm}$$

then

$$\frac{h_{\max}}{h_{\min}} = 1.4$$

when $z = 42$ cm approximately, which is in reasonable agreement with observation.

In addition to the tube inclination effects described here, there is a further well-known cause of film breakdown, namely, the growth of wavy disturbances. A detailed discussion of this is beyond the scope of the present work, but it is not difficult to see that if the wave amplitude is small enough for a linearized theory to be applicable and if α is small also, then the two effects will simply add without interaction.

NOTATION

r, θ, z = cylindrical polar coordinates
w, v, u = corresponding velocity components
$\overline{u}, \overline{v}$ = mean values of u, v
h = film thickness
h_0 = initial film thickness

H = average of h_0
g = gravitation constant
p = pressure
T = surface tension
R = tube radius
C_1, C_2 = dummy constants

Greek Letters

α = tube inclination to vertical
ν = kinematic viscosity
η = similarity variable = $\cot \theta/2 \exp(z \tan \alpha/R)$
ρ = density

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Effect of a Soft Impeller Coating on the Net Formation of Secondary Nuclei

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Recent work has confirmed the importance of secondary nucleation as the principal seed source in continuous crystallizers of the mixed-magma type (Clontz and McCabe, 1971; Johnson et al., 1972; Larson and Bauer, 1972; Randolph and Cise, 1972; Youngquist and Randolph, 1972; Cayey and Estrin, 1967; Ottens and DeJong, 1973; Bennet et al. 1973). These studies indicate that a dominant mechanism causing secondary nucleation is the high speed collision of macro-sized parent crystals with the vessel impeller or circulating pump impeller. Further, the number of crystals formed per impact appears to increase with parent crystal size and weight (proportional to impact stress). In a recent publication, Shah, McCabe, and Rousseau (1973) show that total nuclei formation is reduced in a mixed vessel when the agitator is constructed of a relatively soft material. In their experiment a single $MgSO_4 \cdot 7H_2O$ crystal was allowed to tumble in the vessel and the nuclei formed by collisions with the impeller were counted after a given time in the batch-operated vessel.

In the present experiments the effect of agitator coating on the formation of secondary nuclei in a continuous flow mixed-magma suspension was measured. The system studied was water/K_2SO_4. Identical runs were made and compared using a four-bladed stainless steel impeller before and after covering with 0.76 mm (0.03 in.) of a soft rubber coating of Uralane 8267. Similar runs were made with an aluminum impeller before and after coating with 0.10 mm (0.004 in.) of Teflon.

APPARATUS

The system used for this study consisted of a one-liter glass vessel with draft tube agitation provided by a variable speed axial flow impeller. Solution at a controlled saturation is continuously fed to the vessel at a known rate. Total supersaturation is controlled by controlling vessel temperatures. Under the short retention conditions of the test, supersaturation relief is negligible and suspension supersaturation is mainly determined by the ΔT drop from feed to crystallizer. Dry K_2SO_4 seed crystals of a known weight and size are introduced at the initial run time through an access port; these parent crystals are totally retained by a fine mesh (about 37μ) retaining screen through which the effluent discharges by gravity flow. A fine distribution of secondary nuclei is measured in the effluent using a multichannel Coulter particle counter. The operation of the apparatus is described elsewhere in detail (Cise and Randolph, 1972; Randolph and Cise, 1972).

RESULTS

The population data from the Coulter counter were processed as follows. Background-corrected population density data at a given crystal size were time smoothed with a second-order polynomial fit using a linear regression program. These smoothed data were then plotted on a semi-log plot versus particle size at specific run times. These data were again smoothed in the size domain with another polynomial fit. Figure 1 shows the smoothed data at identical run times of 40 min. for both the stainless and rubber-coated impellers in otherwise similar runs. The data points shown represent the original population mea-